Solutions

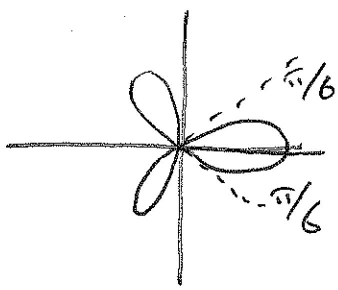
Exam 3
Chapters 15 and 16

Name:________________________

Do not write your name on any other page. Answer the following questions. *Answers without proper evidence of knowledge will not be given credit.* Make sure to make reasonable simplifications. Do not approximate answers. Give exact answers. **No calculators are allowed on this exam.**

Show your work!

1. Find the area of the region inside one loop of the rose $r = \cos 3\theta$. (15 points)
Hint: Recall $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$.



$$\int_{-\pi/6}^{\pi/6} \int_0^{\cos 3\theta} r \, dr \, d\theta$$

$$\int_0^{\cos 3\theta} r \, dr = \frac{\cos^2 3\theta}{2}$$

$$\frac{1}{2}\int_{-\pi/6}^{\pi/6} \cos^2 3\theta = \frac{1}{4}\int_{-\pi/6}^{\pi/6} 1 + \cos 6\theta \, d\theta$$

$$= \frac{1}{4}\left(\theta + \frac{1}{6}\sin 6\theta\right)\Big|_{-\pi/6}^{\pi/6}$$

$$= \frac{\pi}{12}.$$

2. The joint density function for a pair of random variables X and Y is given by

$$f(x,y) = \begin{cases} Cx(1+y) & 0 \le x \le 1, \quad 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of the constant C and find $P(X+Y \le 1)$. (15 points)

$$1 = C\int_0^1 \int_0^2 x(1+y)\,dy\,dx$$

$$\int_0^2 x(1+y)\,dy = 4x$$

$$\int_0^1 4x\,dx = 2 \Rightarrow 2C = 1 \Rightarrow C = \frac{1}{2}$$

$$P(X+Y \le 1) = \frac{1}{2}\int_0^1 \int_0^{1-x} x(1+y)\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 \frac{x(2-x)^2 - x}{2}\,dx$$

$$= \frac{1}{4}\int_0^1 x^3 - 4x^2 + 3x\,dx$$

$$= \frac{1}{4}\left(\frac{3}{2} - \frac{4}{3} + \frac{1}{4}\right) = \frac{5}{24}.$$

3. Evaluate $\iiint_T xyz\, dV$ where T is the solid tetrahedron which lies under the plane $x - y - z = 0$ and is bounded by the xz-plane, xy-plane and the plane $x = 1$. (15 points)

$$\iiint_T xyz\,dV = \int_0^1 \int_0^x \int_0^{x-y} xyz\,dz\,dy\,dx$$

$$\int_0^{x-y} xyz\,dz = \frac{xy(x-y)^2}{2} = \frac{x^3y - 2x^2y^2 + xy^3}{2}$$

$$\frac{1}{2}\int_0^x x^3y - 2x^2y^2 + xy^3\,dy = \frac{1}{2}\left(\frac{x^5}{2} - \frac{2}{3}x^5 + \frac{x^5}{4}\right) = \frac{x^5}{24}$$

$$\int_0^1 \frac{x^5}{24}\,dx = \frac{1}{144}.$$

4. Use spherical coordinates to find the volume inside the region E which is bounded above by $x^2 + y^2 + z^2 = 9$ and bounded below by $x^2 + y^2 + z^2 = 4$ and the xy-plane. (15 points)

$$\int_0^{\pi/2} \int_0^{2\pi} \int_2^3 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$\int_2^3 \rho^2 \sin\phi \, d\rho = \left(\frac{27}{3} - \frac{8}{3}\right) \sin\phi = \frac{19}{3} \sin\phi$$

$$\int_0^{2\pi} \frac{19}{3} \sin\phi \, d\theta = \frac{38\pi}{3} \sin\phi$$

$$\int_0^{\pi/2} \frac{38\pi}{3} \sin\phi \, d\phi = -\frac{38\pi}{3} \cos\phi \Big|_0^{\pi/2} = \frac{38\pi}{3}$$

5. Find the Jacobian of the transformation given by $x = u/v$, $y = v/w$ and $z = w/u$. (10 points)

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} & 0 \\ 0 & \frac{1}{w} & -\frac{v}{w^2} \\ -\frac{w}{u^2} & 0 & \frac{1}{u} \end{vmatrix} = \frac{1}{v}\begin{vmatrix} \frac{1}{w} & -\frac{v}{w^2} \\ 0 & \frac{1}{u} \end{vmatrix} - \left(\frac{-u}{v^2}\right)\begin{vmatrix} 0 & -\frac{v}{w^2} \\ -\frac{w}{u^2} & \frac{1}{u} \end{vmatrix} + 0$$

$$= \frac{1}{v}\left(\frac{1}{wu}\right) + \frac{u}{v^2}\left(-\frac{vw}{w^2u^2}\right) = \frac{1}{uvw} - \frac{1}{uvw} = 0$$

6. Determine whether or not $\mathbf{F}(x, y) = -e^x \cos y\ \mathbf{i} + e^x \sin y\ \mathbf{j}$ is a conservative vector field. If so, find a potential function f such that $\mathbf{F} = \nabla f$. (10 points)

$$\frac{\partial}{\partial y}(-e^x \cos y) = e^x \sin y$$

$$\frac{\partial}{\partial x}(e^x \sin y) = e^x \sin y \quad \text{so yes conservative.}$$

$$\int -e^{+x} \cos y\, dx = -e^x \cos y + C_1(y)$$

$$\int e^x \sin y\, dy = -e^x \cos y + C_2(x)$$

So $f = -e^x \cos y$.

7. Use the results of the previous exercise to evaluate the $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve C where C is given by $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j}$, $(0 \le t \le \pi)$ and $\mathbf{F}(x, y) = -e^x \cos y \; \mathbf{i} + e^x \sin y \; \mathbf{j}$. (10 points)

By Fund. Thm. of Line Integrals

$$\int_C F \cdot dr = f(r(\pi)) - f(r(0))$$

$$= -e^{\pi} \cos(2\pi) - (-e^{0} \cos(0))$$

$$= -e^{\pi} + 1 = 1 - e^{\pi}.$$

8. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ along the curve C given by $\cos t\mathbf{i}+\sin t\mathbf{j}+|t-\pi|\mathbf{k}$ given that $\mathbf{F}$ is a conservative vector field. (10 points)

C is a closed curve on a conservative vector field.

So

$$\int_C F\cdot dr = 0.$$